Algebraic Linear Orderings

Zoltán Ésik[1]

Joint work with:

Stephen L. Bloom[2]

[1]**University of Szeged, Hungary**

[2]**Stevens Institute, Hoboken, NJ**

A regular system

$$\begin{aligned} X &= X + Y + X \\ Y &= \mathbf{1} + Y \end{aligned}$$

Simplest solution:

$$\begin{aligned} X &= \mathbb{N} \times \mathbb{Q} \\ Y &= \mathbb{N} \end{aligned}$$

An algebraic system

$$\begin{aligned} X &= Y(\mathbf{1}) \\ Y(x) &= Z(x) + Y(\mathbf{1} + x) \\ Z(x) &= Z(x) + x + Z(x) \end{aligned}$$

First component of the simplest solution:

$$X = \sum_{n>0} \mathbf{n} \times \mathbb{Q}$$

Outline

Linear orderings

Continuous categorical algebras

Recursion schemes, regular and algebraic objects

Regular and algebraic linear orderings

Conclusion and open problems

Linear Orderings

Linear orderings

Linear ordering: $(P,<)$ where P is a *countable* set and $<$ is a strict linear order relation on P. A **morphism** of linear orderings is an order preserving map. Isomorphic linear orderings have the same **order type**: $\mathbf{o}((P,<))$ or just $\mathbf{o}(P)$

Definition Let $A = \{a_1 < \ldots < a_n\}$ be an ordered alphabet. The lexicographic order on A^* is defined by:

$$x <_\ell y \iff y = xz \text{ for some } z \in A^+,$$
$$\text{or } x = ua_iv,\ y = ua_jw \text{ for some } u,v,w \in A^* \text{ and } a_i < a_j$$

Proposition Every (recursive) linear ordering is isomorphic to the lexicographic ordering $(L,<_\ell)$ of some (recursive) prefix language L (over the binary alphabet $\{0,1\}$).

Examples $(1^*0,<_\ell) \cong (\mathbb{N},<)$ $\quad ((00+11)^*10,<_\ell) \cong (\mathbb{Q},<)$

Linear orderings and trees

Let Σ be a (finite) ranked alphabet. A **Σ-tree** T is defined as a partial function

$$T : \mathbb{N}^* \quad \to \quad \Sigma$$

such that dom(T) is prefix closed and whenever $T(ui)$ is defined for some $u \in \mathbb{N}^*$ and $i \in \mathbb{N}$ then $T(u) \in \Sigma_n$ for some n with $i < n$.

Definition The **frontier** or **leaf ordering** of T is:

$$\mathbf{Fr}(T) = \{u \in \mathbb{N}^* : T(u) \in \Sigma_0\} \subseteq \{0, \ldots, r-1\}^*$$

where $r = \max\{n : \Sigma_n \neq \emptyset\}$.

Proposition Every linear ordering is isomorphic to the leaf ordering of a (binary) tree.

Operations on linear orderings

Sum: $P + Q = P \times \{0\} \cup Q \times \{1\}$, $(x,i) < (y,j)$ iff $i < j$ or $i = j$ and $x < y$.

Generalized sum: $\sum_{x \in P} Q_x = \bigcup_{x \in P} Q_x \times \{x\}$ with $(y,x) < (y',x')$ iff $x < x'$, or $x = x'$ and $y < y'$.

Product: $Q \times P = \sum_{x \in P} Q_x$ where $Q_x = Q$ for all x.

Geometric sum: $P^* = \sum_{n \geq 0} P^n$

Reverse: $-P$

Scattered and dense linear orderings

Definition A linear ordering $(L, <)$ is **dense** if it has at least 2 elements and for any $x < y$ in L there exists $z \in L$ with $x < z < y$. A linear ordering is **scattered** if it has no dense subordering. A linear ordering is a **well-ordering** if any nonempty subset has a least element.

Every well-ordering is scattered. Up to isomorphism, there are 4 countable dense linear orderings:

$$\mathbb{Q}, \quad \mathbf{1} + \mathbb{Q}, \quad \mathbb{Q} + \mathbf{1}, \quad \mathbf{1} + \mathbb{Q} + \mathbf{1}$$

Hausdorff rank

Theorem (Hausdorff) A linear ordering P is scattered iff it belongs to VD_α for some (countable) ordinal α:

$$VD_0 = \{\mathbf{0}, \mathbf{1}\} \quad VD_\alpha = \{\sum_{n \in \mathbb{Z}} P_n : P_n \in \bigcup_{\beta < \alpha} VD_\beta\}$$

The least ordinal α such that $P \in VD_\alpha$ is called the **Hausdorff rank** of the scattered linear ordering P.

Rank 1: $\mathbf{2}, \mathbf{3}, ..., \omega, -\omega, -\omega + \omega$

Rank 2: $\omega + \mathbf{1}, \omega + \omega, \omega \times \mathbf{n}$ $(n \geq 2), \omega^2, \omega + (-\omega)$

Rank 3: $\omega^2 + \mathbf{1}, \omega^2 + \omega, \omega^3$

Rank ω: ω^ω

Theorem (Hausdorff) Every linear ordering is either scattered or a dense sum of scattered linear orderings.

Continuous categorical algebras

Categorical algebras

Categorical Σ-algebra: a small category A together with a collection of functors $\sigma^A : A^n \longrightarrow A$, for each $\sigma \in \Sigma_n$. **Morphisms** of categorical algebras: functors preserving the operations up to natural isomorphism:

$$\begin{array}{ccc} A^n & \xrightarrow{\sigma^A} & A \\ {\scriptstyle h^n}\downarrow & & \downarrow{\scriptstyle h} \\ B^n & \xrightarrow[\sigma^B]{} & B \end{array}$$

Ordered Σ-algebra: the category is a poset and the operations are monotonic. Morphisms are order preserving homomorphism.

Continuous categorical algebras

Continuous categorical Σ-algebra: C has initial object and colimits of ω-diagrams. The operations σ^C preserve colimits of ω-diagrams. **Morphisms** preserve initial object and colimits of ω-diagrams.

Continuous ordered algebra: Continuous categorical Σ-algebra whose underlying category is poset.

Examples of continuous categorical algebras

Examples 1. For any Σ, the Σ-algebra T_Σ of all finite and infinite Σ-trees is a continuous ordered Σ-algebra: Initial continuous categorical Σ-algebra

$$T \leq T' \;\Leftrightarrow\; T(u) = T'(u) \text{ for all } u \in \mathrm{dom}(T)$$

$$\sigma(T_1, \ldots, T_n) = T \;\Leftrightarrow\; T(u) = \begin{cases} \sigma & u = \epsilon \\ T_i(v) & u = iv \text{ where } i \in \mathbb{N}, v \in \mathbb{N}^* \\ \text{undef} & \text{otherwise} \end{cases}$$

Examples of continuous categorical algebras

2. The category $\mathbf{Lin}$ of linear orderings $(P, <)$ is a continuous categorical Δ-algebra, where Δ contains a binary symbol $+$ denoting **sum** and the constant $\mathbf{1}$.

$$\begin{array}{ccccc} P & \longrightarrow & P + Q & \longleftarrow & Q \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f+g} & & \downarrow{\scriptstyle g} \\ P' & \longrightarrow & P' + Q' & \longleftarrow & Q' \end{array}$$

The (essentially) **unique** morphism $T_\Delta \longrightarrow \mathbf{Lin}$ maps a tree T to its frontier $\mathbf{Fr}(T)$.

Recursion schemes, regular and algebraic objects

Initial fixed points

Suppose that $F : C \longrightarrow C$ is an endofunctor on a category C. An F-**algebra** is a morphism $f : Fc \longrightarrow c$. F-algebras form a category:

$$\begin{array}{ccc} Fc & \xrightarrow{\ f\ } & c \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle h} \\ Fd & \xrightarrow[\ g\ ]{} & d \end{array}$$

Lemma (Lambek) If $f : Fc \longrightarrow c$ is an initial F-algebra, then f is an isomorphism.

An **initial fixed point** of F is the object part of an initial F-algebra. (It is unique up to isomorphism.)

Theorem (Adamek,Wand) Suppose that C has initial object and colimits of ω-diagrams. If $F : C \longrightarrow C$ is continuous (i.e., F preserves colimits of ω-diagrams), then there is an initial F-algebra.

Recursion schemes, defined

Over continuous ordered algebras, one can solve recursion schemes by **least fixed points**. Over continuous categorical algebras, one can solve recursion schemes by **initial fixed points**.

Recursion scheme (Nivat, Guessarian, Courcelle ...) E over Σ:

$$\begin{aligned} F_1(x_1, \ldots, x_{k_1}) &= t_1 \\ &\vdots \\ F_n(x_1, \ldots, x_{k_n}) &= t_n \end{aligned}$$

where each t_i is a **term** built from the letters in Σ, the variables $x_1, \ldots, x_{k_i}$ and the **new function variables** $F_1, \ldots, F_n$. A recursion scheme E is **regular** if $k_1 = \ldots = k_n = 0$.

When A is a continuous categorical Σ-algebra, E determines a continuous endofunctor E_A on

$$[A^{k_1} \longrightarrow A] \times \ldots \times [A^{k_n} \longrightarrow A]$$

and thus has an **initial** fixed point. We let $E_A^\dagger$ denote this initial solution of E over A.

Algebraic and regular objects

Definition An **algebraic functor** $G : A^k \longrightarrow A$ over a continuous categorical Σ-algebra A is any component of $E_A^\dagger$, for some scheme E. When $k = 0$, we identify G with an object of A, called an **algebraic object**. When E is regular, we call G a **regular object**.

Regular objects in T_Σ: **regular trees**

Regular objects in $\mathbf{Lin}$: **regular linear orderings**

Algebraic objects in T_Σ: **algebraic trees**

Algebraic objects in $\mathbf{Lin}$: **algebraic linear orderings**

Branch languages of regular and algebraic trees

The branch language of a tree $T \in T_\Sigma$:

$$\mathbf{Br}(T) = \{uT(u) : u \in \mathsf{dom}(T)\}$$

Theorem (Courcelle, Ginali, Elgot-Bloom-Wright) A tree in T_Σ is regular iff its branch language is regular.

Theorem (Courcelle) A tree in T_Σ is algebraic iff its branch language is a dcfl.

An example

$$
\begin{aligned}
F_0 &= F(a) \\
F(x) &= f(x, F(g(x)))
\end{aligned}
$$

```
T  =   f
      / \
     a   f
        / \
       g   f
       |  / \
       a g   ...
         |
         g
         |
         a
```

$$\mathbf{Br}(T) = \{1^n 0^{n+1} a,\ 1^n 0^m g,\ 1^n f : n \geq 0,\ n \geq m > 0\}$$

A Mezei-Wright theorem

Theorem Suppose that A and B are continuous categorical Σ-algebras and $h : A \longrightarrow B$ is a morphism. Then an object $b \in B$ is regular (algebraic, resp.) iff b is isomorphic to $h(a)$ for some regular (algebraic, resp.) object $a \in A$.

Corollary A linear ordering is regular (algebraic, resp.) iff it is isomorphic to the frontier of a regular (algebraic, resp.) tree over Δ (or any ranked alphabet).

Using the characterization of regular and algebraic trees by branch languages:

Proposition A linear ordering is regular (algebraic) iff it is isomorphic to the lexicographic ordering of a (prefix-free) regular language (det. context-free language) (over the binary alphabet {0,1}).

Examples

$\mathbf{n}\ (n \geq 2)$

```
n =      +
        / \
       1   +
          / \
         1   ...
               \
                +
               / \
              1   1
```

$L_n = \{0, 10, \ldots, 1^{n-1}0, 1^n\}$

$\mathbb{Q}$

```
Q =      +
        / \
       Q   +
          / \
         1   Q
```

$L_Q = (0 + 11)^{*}10$

$\mathbf{1} + \mathbb{Q} + \mathbf{2} + \mathbb{Q} + \ldots$

```
T =      +
        / \
       1   +
          / \
         Q   +
            / \
           2   +
              / \
             Q   ...
```

$L_T = (11)^{*}10L_Q + \sum_{n \geq 0} 1^{2n}0L_n$

Regular and algebraic linear orderings

Heilbrunner's theorem

Theorem (Heilbrunner) A linear ordering is regular iff it can be generated from $\mathbf{0}$ and $\mathbf{1}$ by the $+$,

$$P \mapsto P \times \mathbb{N}, \quad , P \mapsto P \times (-\mathbb{N})$$

and the n-ary **shuffle operations**

$$(P_1, \ldots, P_n) \mapsto \eta(P_1, \ldots, P_n),$$

for all $n \geq 1$.

These operations are the **regular operations**.

Corollary A scattered linear ordering is regular iff it can be generated from $\mathbf{0}, \mathbf{1}$ by the $+$,

$$(-) \times \mathbb{N}, \quad (-) \times (-\mathbb{N})$$

operations. A well-ordering is regular iff it can be generated from $\mathbf{0}, \mathbf{1}$ by the $+$ and $(-) \times \mathbb{N}$ operations.

Corollary The Hausdorff rank of any scattered regular linear ordering is finite.

Corollary A well-ordering is regular iff its order type is $< \omega^\omega$.

Regular linear orderings also appear in the work of Läuchli and Leonard:

$$\forall P \ \forall n \ \exists R \ \textbf{regular} \ P \approx_n R$$

Axiomatization

Some valid equations:

$$
\begin{aligned}
(x+y)\times\mathbb{N} &= x+(y+x)\times\mathbb{N}\\
(x+x)\times(-\mathbb{N}) &= x\times(-\mathbb{N})\\
\eta(x,y) &= \eta(y,x)\\
\eta(x,y)+x+\eta(x,y) &= \eta(x,y)\\
\eta(x)\times\mathbb{N} &= \eta(x)\\
\eta(\eta(x,y),x) &= \eta(x,y)
\end{aligned}
$$

Theorem The equational theory of (regular) linear orderings equipped with the regular operations is decidable in Ptime and can be axiomatized by an infinite number of natural equations.

For $+$ and $(-)\times\mathbb{N}$: Bloom and Choffrut.

Scattered algebraic orderings

Theorem The Hausdorff rank of any scattered algebraic linear ordering is $< \omega^\omega$.

The proof uses prefix grammars:

Definition A **prefix grammar** is a cfg $G = (N, \{0, 1\}, R, S)$ such that for each nonterminal X, $(L(G, X), <_\ell)$ is a prefix language.

Proposition For every recursion scheme over Δ defining an algebraic tree T there is a prefix grammar G such that $\mathbf{Fr}(T) = L(G)$. Moreover, G can be constructed in Ptime.

Theorem If $(L(G), <_\ell)$ is a scattered linear ordering for a prefix grammar G with n nonterminals, then the rank of $(L(G), <_\ell)$ at most $\omega^{n-1} + 1$.

Algebraic well-orderings

Theorem A well-ordering is algebraic iff its order type is $< \omega^{\omega^\omega}$.

An equivalent condition is: a well-ordering is algebraic iff its Hausdorff rank is $< \omega^\omega$. Thus, it suffices to prove:

Every ordinal less than ω^{ω^ω} is algebraic.

Algebraic well-orderings

Proposition The least set of ordinals containing $0, 1$ and closed under $+$, $\times$ and the $^{\omega}$**-power operation** $\alpha \mapsto \alpha^{\omega}$ is the set of ordinals $< \omega^{\omega^{\omega}}$ (i.e., the ordinal $\omega^{\omega^{\omega}}$).

Proposition Algebraic linear orderings contain $0, 1$ and are closed under $+$, $\times$, and geometric sum. Thus, algebraic ordinals are closed under the $^{\omega}$-power operation, since if $\alpha > 1$ then

$$\alpha^{\omega} = \alpha^{*} = \sum_{n \geq 0} \alpha^{n}$$

Closure properties

Either trees or dcfl's can be used. Let $L, L' \subseteq \{0,1\}^*$ such that

$$(L, <_\ell) \cong (P, <) \quad \text{and} \quad (L', <_\ell) \cong (Q, <)$$

Sum: $(0L \cup 1L', <\ell) \cong P + Q$. Moreover, if L, L' are dcfl's, then so is $0L \cup 1L'$.

Product: Suppose that L, L' are *prefix* languages. Then $(L'L, <_\ell) \cong P \times Q$. Moreover, if L, L' are dcfl's, then so is $L'L$.

Geometric sum: Suppose that L is a *prefix* language. Then

$$(\bigcup_{n \geq 0} 1^n 0 L^n, <_\ell)$$

is isomorphic to $P^* = \sum_{n \geq 0} P^n$. Moreover, if L is a dcfl, then so is $\bigcup_{n \geq 0} 1^n 0 L^n$.

Reversal: Reverse the ordering of the alphabet. **Intervals**: Dcfl's are closed under intersection with regular sets. **Regular ops** ...

Example $L = 1^*0$, so that $\mathbf{o}(L, <_\ell) = \omega$.

$$G: \quad S \longrightarrow 0|1SX \qquad X \longrightarrow 0|1X$$

Then

$$L(G) = \{1^n 0 (1^*0)^n : n \geq 0\}$$

$$\mathbf{o}(L(G), <_{\text{lex}}) = \sum_{n \geq 0} \omega^n = \omega^\omega$$

Some decision problems

Theorem It is decidable in polynomial time whether the lexicographic ordering of the language generated by a prefix grammar is scattered, or a well-ordering.

Corollary It is decidable in polynomial time whether the algebraic linear ordering defined by a recursion scheme is scattered, or a well-ordering.

Remark (Luc Boasson) There exists no algorithm to decide for a context-free grammar G (over $\{0, 1\}$) whether or not $L(G)$ is a prefix language, or G is a prefix grammar.

Some decision problems

Theorem It is undecidable for a context-free (prefix) grammar G (over $\{0,1\}$) whether or not $(L(G), <_\ell)$ is dense.

Theorem It is undecidable for a context-free (prefix) grammar G (over $\{0,1\}$) whether or not $(L(G), <_\ell)$ is regular.

Theorem It is undecidable for a context-free (prefix) grammars G_1, G_2 (over $\{0,1\}$) whether or not $(L(G_1), <_\ell)$ is isomorphic to $(L(G_2), <_\ell)$.

Summary

Algebraic and regular objects in cca's, Mezei-Wright theorem.

Algebraic (regular) linear orderings can be represented as frontiers of algebraic (or regular) trees, or as lexicographic orderings of dcfl's (or regular languages).

Regular well-orderings are those well-orderings of order type $< \omega^\omega$ (= **automatic ordinals**, Delhommé)

Scattered regular orderings have finite Hausdorff-rank.

Algebraic well-orderings are those well-orderings of order type $< \omega^{\omega^\omega}$ (= **tree automatic ordinals**, Delhommé)

The Hausdorff-rank of a scattered algebraic linear ordering is $< \omega^\omega$.

It is decidable in Ptime whether an algebraic linear ordering is scattered or a well-ordering.

Some questions

Does there exist a "context-free" linear ordering that is not deterministic?

Is it decidable whether two algebraic linear orderings are isomorphic? (regular case: Thomas, Bloom–Ésik)

Find an operational characterization of scattered algebraic linear orderings.

A hierarchy of schemes was defined by Damm, Gallier and others. Level 0: regular schemes, Level 1: algebraic schemes, Level 2: hyper-algebraic schemes, ...

Some conjectures

Conjecture 1. A well-ordering is definable by a level n scheme iff its order type is $<\Uparrow(\omega, n+2)$. 2. If a scattered linear ordering is definable by a level n scheme then its Hausdorff rank is $<\Uparrow(\omega, n+1)$.

Remark By an unpublished paper of Laurent Braud, any ordinal less than $\Uparrow(\omega, n+2)$ is definable by a level n recursion scheme.

Conjecture 1. A well-ordering is definable by a scheme of some level iff its order type is $< \epsilon_0$. 2. If a scattered linear ordering is definable by a recursion scheme of some level then its Hausdorff rank is $< \epsilon_0$.

Thus, the conjecture is that the ordinals definable by higher order recursion schemes are exactly the ordinals less than the proof theoretic ordinal of Peano arithmetic ...

How about ordinals or scattered linear orderings in Caucal's hierarchy?